# FURTHER MATHEMATICS

**Paper 9231/01**

**Paper 1**

## General comments

There were a few scripts of outstanding quality and a similar number of very poor scripts. The majority of candidates displayed a pleasing level of competence, but found difficulty with the more challenging parts of the paper. Work was generally well presented and it was easy to follow candidates' reasoning. Algebraic manipulation was sound and numerical accuracy was, for the most part, good.

There was no evidence to suggest that candidates had any difficulty completing the paper in the time allowed. A very high proportion of scripts had substantial attempts at all eleven questions. Once again there were few misreads and few rubric infringements.

It was felt that candidates had a sound knowledge of most topics on the syllabus. As in many previous papers, work on vectors and complex numbers caused difficulty for candidates. Question 11 involved an alternative on finite series, involving proof by induction, and an alternative on eigenvalues and eigenvectors. The former proved to be much more popular.

## Comments on specific questions

### Question 1

Something approaching half of the candidates did not appreciate that the integral of $\frac{dy}{dx}$ was $y$ and, likewise, the integral of $\frac{d^2y}{dx^2}$ was $\frac{dy}{dx}$. Consequently there was much unnecessary differentiation and integration. Those who did not fall into this category invariably scored full marks on this question quite rapidly. The others frequently made some manipulative error and lost one, or both, accuracy marks.

*Answer*: **(ii)** 2.

### Question 2

Many candidates did not know how to begin this question. Those who knew that they needed to find the vector perpendicular to the lines *AB* and *OC*, using a cross-product, sometimes made sign errors which destroyed further accuracy. A substantial number could not progress beyond finding the appropriate cross-product. Of those who did, some did not make the common perpendicular a unit vector, while others did not have an appropriate vector to dot with the unit normal vector. The number of candidates successfully completing the question was disappointingly small.

*Answer*: $-1$.

### Question 3

This question was well done by the vast majority of candidates. Nearly all could find the correct intersections with the coordinate axes and the equations of the asymptotes. The most likely reasons for loss of marks were either not showing the forms of the graph at infinity, or omitting to show the lower branch of the curve, sometimes because the graph went off the bottom of the candidate's sheet of paper.

*Answers*: **(i)** (1,0), (4,0), (0,4); **(ii)** $x = -1$, $y = x - 6$ .

**Question 4**

Most candidates were able to find a correct expression for $\frac{dy}{dx}$. Many weaker candidates then simply found $\frac{d}{dt}\left(\frac{dy}{dx}\right)$; only the better candidates realised that they required either $\frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$ or $\frac{\dot{y}\ddot{x}-\dot{x}\ddot{y}}{(\dot{x})^3}$ for $\frac{d^2y}{dx^2}$. Very few candidates offered a satisfactory explanation for the final part of the question. Where something was offered, it usually consisted of considering three isolated points, without considering what might occur between them. It was envisaged that candidates would consider the overall sign of the second derivative in the ranges $(-\pi, 0)$ and $(0, \pi)$, getting $\frac{(-)(-)}{(+)}$ and $\frac{(+)(+)}{(+)}$ respectively, hence a positive result in each case, implying that $\frac{dy}{dx}$ was an increasing function over the range $(-\pi, \pi)$. A graph of either $\frac{dy}{dx}$ or $\frac{d^2y}{dx^2}$ obtained from a graphical calculator could also provide sufficient reasoning.

*Answer*: $\frac{dy}{dx}=\frac{2t-2\sin t}{1+\cos t}$.

**Question 5**

Almost all candidates were able to obtain the cubic equation in $y$. Very few candidates could offer a complete verification that the roots of this equation were $\beta\gamma$, $\gamma\alpha$ and $\alpha\beta$. The most usual response was to indicate that the sum of the roots in the $y$ equation was $-(-5)=5$, which was $\sum\alpha\beta$ in the $x$ equation. Only a small minority showed that the value of $\sum(\alpha\beta)(\beta\gamma)$ was zero and the value of $(\alpha\beta)(\beta\gamma)(\gamma\alpha)$ was 9. Only the most able candidates were able to say that since $\alpha\beta\gamma=-3$ then $\alpha\beta=\frac{-3}{\gamma}$, $\beta\gamma=\frac{-3}{\alpha}$ and $\gamma\alpha=\frac{-3}{\beta}$ hence the equation in $y$ had roots $\alpha\beta, \beta\gamma \text{ and } \gamma\alpha$. Most candidates tackled the final part of the question reasonably efficiently. Errors included not squaring $\sum\alpha$ when using $\sum\alpha^2=\left(\sum\alpha\right)^2-2\sum\alpha\beta$ or misquoting a result such as $\sum\alpha^3=\sum\alpha\left(\sum\alpha^2-\sum\alpha\beta\right)+3\alpha\beta\gamma$ for the second result. Substituting each root into the equation and summing was generally a far more satisfactory approach for the second result. Mistakes crept in when summing the constant term however.

*Answers*: $y^3-5y^2-9=0$, 25, 152.

**Question 6**

Most candidates were able to differentiate the given expression correctly. Weaker candidates were not able to manipulate it into the required form stated in the question. The majority of candidates were able to use the initial result to obtain the reduction formula. Nearly all candidates were able to make a good attempt at using the reduction formula, even if they had been unsuccessful earlier in the question. The failure to use brackets correctly sometimes resulted in an inaccurate final answer.

*Answer*: $I_4=\pi-\frac{7\sqrt{3}}{4}$.

**Question 7**

Better candidates were able to tackle this question competently, using the result $(z - z^{-1})^6 = (z^6 - z^{-6}) - 6(z^4 - z^{-4}) + 15(z^2 - z^{-2}) - 20$. Some errors occurred with signs, usually due to mistakenly thinking that $\mathrm{i}^6$ was 1, rather than $-1$. Incorrect binomial coefficients were another source of error. The occasional candidate was sufficiently adept with double and triple angle formulae to obtain the correct result from the binomial expansion of $(\cos\theta + \mathrm{i}\sin\theta)^6$, but this was not a recommended approach and candidates frequently made little, or no, progress. Integration of the result was usually completed correctly.

*Answers*: $\frac{5}{16} - \frac{15}{32}\cos 2\theta + \frac{3}{16}\cos 4\theta - \frac{1}{32}\cos 6\theta$; $\frac{5}{64}\pi$.

**Question 8**

There were many completely correct solutions to both parts of this question. In part **(a)**, some weaker candidates correctly obtained $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin x}{\cos x}$ but failed to write it as tan *x*, while others failed to use $\sec^2 x = 1 + \tan^2 x$. Some, who successfully found the integrand to be sec *x*, could not integrate it, despite the result being in list of formulae (MF10). In part **(b)** some weaker candidates, although knowing the correct formula, could not simplify the integrand to $(x+4)^{\frac{1}{2}}$, while others, who could, sometimes integrated incorrectly.

**Question 9**

The proof of the initial result caused considerable difficulty to many candidates. One way of proceeding was to say: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}u}$ since $\frac{\mathrm{d}x}{\mathrm{d}u} = \mathrm{e}^u = x$, hence $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{1}{x^2}\frac{\mathrm{d}y}{\mathrm{d}u} + \frac{1}{x}\frac{\mathrm{d}^2y}{\mathrm{d}u^2} \times \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x^2}\left(\frac{\mathrm{d}^2y}{\mathrm{d}u^2} - \frac{\mathrm{d}y}{\mathrm{d}u}\right)$, from which the result follows. It was necessary to consider the general case and not to consider *y* as some specific function of *x*, as a small number of candidates did. Rather more candidates could derive the differential equation, by using the printed result from the first part and, at least, realising that $x\frac{\mathrm{d}y}{\mathrm{d}x}$ was $\frac{\mathrm{d}y}{\mathrm{d}u}$. Many were able to find the complementary function and the particular integral, but a sizeable minority lost the last mark by not expressing *y* in terms of *x*.

*Answer*: $y = \frac{A}{x} + \frac{B}{x^3} + 2x^2$.

**Question 10**

Part **(i)** was done well by almost all candidates. There was little success, however, with parts **(ii)** and **(iii)**. The key to these parts was to realise that $y = r\sin\theta = a\sin 3\theta \sin\theta$ and this had to be maximised. In part **(iv)** the sketch was usually basically correct, although a mark was often lost by showing symmetry about either $\theta = \frac{\pi}{4}$ or $\theta = \frac{\pi}{3}$, rather than $\theta = \frac{\pi}{6}$. Some sketches did not return to the pole at $\theta = \frac{\pi}{3}$.

*Answer*: **(iii)** $\frac{9}{16}a$.

**Question 11 EITHER**

This alternative, as stated earlier, was far more popular with candidates. The proof by induction was generally well done, although many did not state an adequate conclusion. Something along the lines of : '$H_1$ is true and $H_k \Rightarrow H_{k+1}$ hence $H_n$ is true for all positive integers *n*', would suffice. The two results which followed were generally obtained by most candidates attempting this alternative, although some lost a mark by not stating the values of *a*, *b* and *μ,* as requested in the question. The final part was poorly done and many candidates thought that merely stating the printed result would acquire the marks. The best way of proceeding was to write $N^{-4}S_N = 5 + \frac{22}{N} + \frac{17}{N^2}$. Then if N ⩾ 18, $0 < \frac{17}{18N} < \frac{1}{N}$ and the result follows.

*Answers*: $\mu = -12$, $a = 5$, $b = 17$.

**Question 11 OR**

Most attempts at this alternative got little further than finding a value for *a* and finding the eigenvectors $\mathbf{e}_1$ and $\mathbf{e}_2$. There were some arithmetical errors in doing both of these things. In part **(i)** only the very assured candidates wrote $\mathbf{x} = p\mathbf{e}_1 + q\mathbf{e}_2 \in V$, where *p* and *q* are constants, hence $\mathbf{Ax} = -2p\mathbf{e}_1 - 5q\mathbf{e}_2 \in V$. In part **(ii)** it was necessary to find $\mathbf{e}_1 \times \mathbf{e}_2$ then show that the resulting column vector, when pre-multiplied by **A,** did not give a parallel column vector to that obtained from the cross-product. A number of candidates, who did not realise this, solved the characteristic equation and found the third eigenvector, which they showed was not parallel to $\mathbf{e}_1 \times \mathbf{e}_2$. This approach involved a considerable amount of work.

*Answer*: **(ii)** $\mathbf{i} - \mathbf{j} + \mathbf{k}$.

# FURTHER MATHEMATICS

**Paper 9231/02**

**Paper 2**

## General comments

The quality of the candidates' work varied greatly so in that sense the paper discriminated well. It did not appear to be too long for the time allowed, since the better candidates were able to make reasonably complete attempts at all questions. In general the Statistics questions were answered more successfully, though some parts defeated most candidates, such as the second part of **Question 10** or part **(iii)** of **Question 11 OR**. Although this latter optional question attracted more attempts than the Mechanics one, its relative popularity was less than last year. As usual some questions were found by the candidates to be more demanding than others, but none appeared to be unduly hard or easy.

## Comments on specific questions

### Question 1

Most candidates appreciated that when the string slackens the centripetal force is opposed only by the component of the weight of the particle, thus allowing its given speed to be shown. Substitution of this speed into the conservation of energy equation yields the initial speed of projection.

*Answer*: $\sqrt{\dfrac{7ga}{2}}$.

### Question 2

This question was answered less well than the previous one, with many candidates wrongly applying equations of linear rather than circular motion. The magnitude of the braking force is best found by first multiplying the magnitude 0.4 of the constant angular deceleration by the moment of inertia $\frac{3}{16}$, and equating this to the couple exerted by the braking force. The angle turned through by the wheel may be found from one of several relevant standard equations of circular motion under constant angular acceleration, or from conservation of energy.

*Answers*: 0.3 N; 5 radians.

### Question 3

By applying conservation of momentum and Newton's restitution equation, the speed $v_A$ of $A$ after its collision with $B$ may be found in terms of $e$ and $u$, and shown to be in the opposite direction to its initial motion since $e > \frac{1}{3}$. While most candidates knew that $B$ would rebound from the barrier with its speed altered by a factor $e$, very few were able to show that this would exceed $v_A$ unless $e = 1$. Some made a reasonable attempt based on considering either the ratio of the two speeds or the relative speed of the particles, apart from introducing inconsistent signs relating to the changes of direction of $A$ and $B$, but many considered only the special case of $e = 1$.

**Question 4**

Substitution of the combined moment of inertia $\frac{11}{3}ma^2$ of the rod and ring into the energy equation yields the required result. While most candidates completed this and the final part satisfactorily, the friction force *F* and normal contact force *R* presented more difficulty. These are found by two resolutions of the forces on the ring, one along and the other normal to the rod respectively. The latter involves the angular acceleration, which is readily found by differentiation of the result given in the first part of the question. The last part follows from the standard result $\mu = \frac{F}{R}$.

*Answer*: $\tan^{-1}\left(\frac{2\mu}{29}\right)$.

**Question 5**

Although the initial taking of moments about *A* for the system was usually performed correctly, the remainder of the question defeated many candidates. A common fault was to overlook the force exerted on the rod *AB* by the pivot at *A*, so that 2*W* was wrongly equated to the vertical component of the tension, for example, while another was to assume that the force acting on the rod *BC* at *B* was in a particular direction, such as along *AB* or *BC*. More generally, many candidates wrote down seemingly incorrect equations without saying what they represented, perhaps in some cases because they were themselves unsure of what they were doing. The tension can be found immediately by taking moments for the rod *B*C about *B*, while the force at *B* requires any two independent moment or resolution equations (out of several possible choices) for its components, followed by their combination.

*Answers*: 0.671*W*; 0.5 W.

**Question 6**

Almost all candidates used the appropriate standard formula for the confidence interval, though quite a number took the sample mean to be 10 instead of calculating 10.03 exactly from the given diameters of the sample discs, and some used an incorrect tabular *t*-value in place of 4.032. The confidence interval for the mean circumference is of course obtained by increasing that for the mean diameter by a factor $\pi$.

*Answers*: (9.97, 10.1); (31.3, 31.7).

**Question 7**

The required probability was often found correctly from $1 - F(\frac{3}{4})$, but few candidates then deduced that the upper quartile must exceed $\frac{3}{4}$. Calculating the upper quartile is not what the question requires. The cumulative distribution function of *Y* is here $P(-\sqrt{y} \leqslant X \leqslant \sqrt{y})$, and hence $F(\sqrt{y}) - F(-\sqrt{y})$, but many of those who adopted this approach found instead $P(X \leqslant \sqrt{y})$ and hence an incorrect result.

*Answers*: $\frac{37}{128}$; 0 ($y < 0$), $y^{\frac{3}{2}}$ ($0 \leqslant y \leqslant 1$), 1 ($y > 1$).

**Question 8**

Few candidates stated that it is unnecessary to assume that the two population variances are equal since the two samples are sufficiently large, or that the samples should be assumed to be random. Most, however, were able to carry out the test satisfactorily, comparing the calculated *z*-value of magnitude 1.92 with the tabular value 1.645 in order to conclude that the controlled grazing mean does indeed exceed the freely grazing one by less than 5 kg.

**Question 9**

This question was often answered well, with the exception of the final deduction. Most candidates noted that the first of the expected values is less than 5, so that the first two cells should be combined. The calculated value 9.22 of $\chi^2$ is then compared with the tabulated value 7.815 in order to conclude that the null hypothesis should be rejected, and thus that the data does not conform to a binomial distribution with the stated parameters. From this it can be deduced that $p \neq 0.6$.

**Question 10**

Many candidates found the probability that the archer requires at least 6 shots to hit the bull's-eye by using a geometric distribution with parameter $\frac{3}{20}$, though some mistakenly employed a binomial distribution. Very few, though, knew how to obtain the given $\mathrm{P}(Y = r)$, which is found most easily as the product of the probability of 2 bull's-eye hits in $r - 1$ shots and the probability of hitting the bull's-eye on the $r$th shot. The simplification of the given ratio of probabilities was usually conducted correctly, and hence the required set of values of $r$ and the most probable value of $Y$ found.

*Answers*: 0.444; $\frac{17r}{20(r-2)}$; $r > 13$; 14.

**Question 11 EITHER**

While this alternative was less popular than the Statistics one, some reasonable attempts were seen. The equilibrium position, where the extension is $\frac{1}{4}l$, is determined by equating the weight of the particle to the tension at that point, and the SHM equation then follows from applying Newton's law at the general point. Since the given initial speed must equal $A\omega$, this enables the amplitude $A = \frac{1}{2}l$ to be found, and application of the standard equation $v^2 = A^2(\omega^2 - x^2)$ produces the required speed $v$ when $x = -\frac{1}{4}l$. Finding the time in the last part requires separate consideration of the simple harmonic motion to the point at which the length of the string is $l$, followed by the motion under gravity to the highest point. The former may be tackled in a variety of ways, using the standard form $x = A \cos \omega t$ or $A \sin \omega t$, while the time for the motion under gravity is simply $\frac{v}{g}$.

*Answer*: $\sqrt{\frac{3gl}{4}}$.

**Question 11 OR**

Although many candidates marked the vertical distances of the points from the regression line on their diagram, very few stated that what is minimised is the sum of the squares of these distances. They also rarely stated the reason why the two regression lines are not the same, namely that the points are not collinear, or some equivalent statement. The value of the product moment correlation coefficient was often calculated correctly, though a satisfactory comment along the lines of the points lying close to a straight line was given relatively infrequently. A comment solely on the correlation of the points is not entirely satisfactory as a description of the scatter diagram. Most candidates appreciated, though, that 0.975 being closer to 1 than 0.965 explained why one regression line is more suitable than the other, and were able to find the equation of the former line. The comments made on the constant term in the equation were rarely enlightening. The Examiners were hoping, for example, that its physical significance would be explained, namely that it represents the value of $\sqrt{z}$ for zero speed, together with a remark such as one would expect it to be zero, and either the fact that it is not perhaps indicates an error in the model or the data or alternatively that it is so small as to be approximately zero.

*Answers*: **(i)** 0.975; **(ii)** $\sqrt{z} = 0.0664v - 0.0207$.